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SYSTEMS OF COORDINATE FUNCTIONS IN HEAT-CONDUCTION PROBLEMS FOR MULTILAYER BODIES

V. A. Kudinov, R. Zh. Gabdushev, V. A. Obukhov, and V. V. Nekrylov

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The authors have investigated the efficiency of using different systems of coordinate functions in heat-conduction problems for piecewise-homogeneous bodies. The analytical solutions obtained are compared with a computer-assisted calculation by numerical methods.

In [1, 2] the authors develop a method of obtaining analytical solutions of boundary-value problems of heat conduction for multilayer structures that is based on the combined use of exact and approximate analytical methods [3]. In this approach, an exact method (Fourier, integral transforms, etc.) is used in relation to the independent parabolic variable, and an approximate methods (variational, weighted discrepancies, etc.) is used in relation to the independent elliptic coordinates. In the indicated works the authors discuss some approaches to constructing systems of coordinate functions that exactly satisfy the boundary conditions and conjugation conditions, including ones using local coordinate systems.

The present work is concerned with investigations of the accuracy of solutions obtained using different systems of coordinate functions. The analytical solutions are compared with solutions obtained by numerical methods.

Mathematically, the formulation of the heat-conduction problem for a two-layer plate using local coordinate systems with boundary conditions of the first kind (a symmetric problem) has the form

$$\frac{\partial \Theta_i(\eta_i, \mathrm{Fo})}{d\mathrm{Fo}} = \frac{\frac{a_i}{a}\,\partial^2 \Theta_i(\eta_i, \mathrm{Fo})}{d\eta_i^2}; \quad (1)$$

$$\Theta_i(\eta_i, 0) = 1; \quad (2)$$

$$\frac{\partial \Theta_1(0, \mathrm{Fo})}{\partial \eta_1} = 0; \quad (3)$$

$$\Theta_1(\Delta_1, \mathrm{Fo}) = \Theta_2(0, \mathrm{Fo}); \quad (4)$$

$$\lambda_1 \frac{\partial \Theta_1(\Delta_1, \mathrm{Fo})}{\partial \eta_1} = \lambda_2 \frac{\partial \Theta_2(0, \mathrm{Fo})}{\partial \eta_2}; \quad (5)$$

$$\Theta_2(\Delta_2, \mathrm{Fo}) = 0. \quad (6)$$

The solution of the problem (1)-(6) in the zeroth and first approximations is sought in the form

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$$\Theta_i(\eta_i, \text{Fo}) = f_1(\text{Fo})\, \varphi_{1i}(\eta_i)\,, \quad i = 1, 2\,, \tag{7}$$

where $f_1(\text{Fo})$ is an unknown function of time, and $\varphi_{1i}(\eta_i)$ are coordinate functions determined by the formulas

$$\varphi_{11}(\eta_1) = A_1 + A_2\eta_1^2\,; \tag{8}$$

$$\varphi_{12}(\eta_2) = A_3 + A_4\eta_2 - \eta_2^2\,, \tag{9}$$

where A_1, A_2, A_3, and A_4 are coefficients determined in such a way that the boundary conditions and the conjugation conditions are satisfied exactly:

$$A_1 = \Delta_2^2 + \Delta_1\Delta_2 + \frac{1}{2}\frac{\lambda_2}{\lambda_1}\Delta_1^2\,;\quad A_2 = -\frac{1}{2}\frac{\lambda_2}{\lambda_1}\,;$$

$$A_3 = \Delta_2^2 + \Delta_1\Delta_2\,;\quad A_4 = -\Delta_1\,.$$

To find the unknown function of time $f_1(\text{Fo})$, we specify the discrepancy of Eq. (1) and require that it be orthogonal to the coordinate function equal to unity (the zeroth approximation [1, 2]). Hence we arrive at the following ordinary differential equation:

$$\dot{f}_1(\text{Fo})\, N - f_1(\text{Fo})\, M = 0\,, \tag{10}$$

where

$$N = \int\limits_0^{\Delta_1} (A_1 + A_2\eta_1^2)\, d\eta_1 + \int\limits_0^{\Delta_2} (A_3 + A_4\eta_2 - \eta_2^2)\, d\eta_2\,;\quad M = \frac{a_1}{a}\, 2\int\limits_0^{\Delta_1} d\eta_1 + \frac{a_2}{a}\, 2\int\limits_0^{\Delta_2} d\eta_2\,.$$

The total integral of Eq. (10) is

$$f_1(\text{Fo}) = C \exp(M\text{Fo}/N)\,, \tag{11}$$

where C is the integration constant, determined from the initial condition (2):

$$C = (\Delta_1 + \Delta_2)/N\,. \tag{12}$$

With account for (11), (12), expression (7) acquires the form

$$\Theta_{1i}(\eta_i, \text{Fo}) = \frac{\Delta_1 + \Delta_2}{N} \exp(M\text{Fo}/N)\, \varphi_{1i}(\eta_i)\,. \tag{13}$$

Without using local coordinate systems the problem (1)-(6) can be mathematically formulated as

$$\frac{\partial\Theta_i(\eta_i, \text{Fo})}{\partial\text{Fo}} = \frac{a_i}{a}\frac{\partial^2\Theta_i(\eta_i, \text{Fo})}{d\eta^2}\,,\quad 0 \le \eta_i \le 1\,,\quad i = 1, 2\,; \tag{14}$$

$$\Theta_i(\eta_i, 0) = 1\,; \tag{15}$$

$$\partial\Theta_1(0, \text{Fo})/\partial\eta = 0\,; \tag{16}$$

$$\Theta_1(\eta_1, \text{Fo} = \Theta_2(\tau_1, \text{Fo})\,; \tag{17}$$

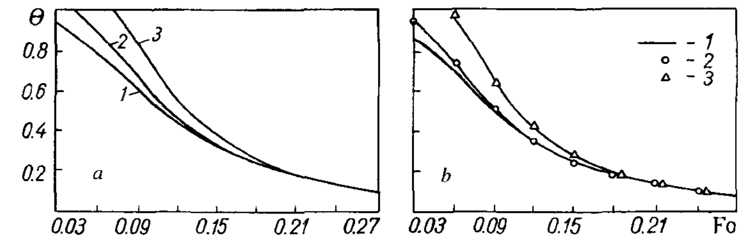

Fig. 1. Distribution of the dimensionless temperature in the two-layer plate at $\eta = 0$ (a) and $\eta = \eta_1$ (b): 1) the method of finite differences; 2) the same coordinate system for both layers, $\eta = 0$; 3) local (separate for each layer) coordinate systems (for a: $\eta_1 = 0$; for b: $\eta_1 = 1$).

$$\lambda_1 \frac{\partial \Theta_1(\eta_1, \mathrm{Fo})}{\partial \eta} = \lambda_2 \frac{\partial \Theta_2(\eta_1, \mathrm{Fo})}{\partial \eta}; \quad (18)$$

$$\Theta(1, \mathrm{Fo}) = 0, \quad (19)$$

where $\eta_i = x_i/\delta$; $\delta = \delta_1 + \delta_2$.

The solution of the problem in the zeroth and first approximations is sought in the form

$$\Theta_{1i}(\eta, \mathrm{Fo}) = f_1(\mathrm{Fo})\, \varphi_{1i}(\eta), \quad i = 1, 2, \quad (20)$$

where coordinate functions $\varphi_{1i}(\eta)$ that exactly satisfy the boundary conditions and the conjugation conditions are determined by the formulas

$$\varphi_{11}(\eta) = B_1 + B_2 \eta^2; \quad (21)$$

$$\varphi_{12}(\eta) = 1 - \eta^2, \quad (22)$$

where

$$B_1 = 1 + \left(\frac{\lambda_2}{\lambda_1} - 1\right)\eta_1^2; \quad B_2 = -\lambda_2/\lambda_1.$$

In this case, the solution of the problem in the zeroth approximation is

$$\Theta_{1i}(\eta, \mathrm{Fo}) = \frac{1}{N_1} \exp(N_2 \mathrm{Fo}/N_1)\, \varphi_{1i}(\eta), \quad (23)$$

where

$$N_1 = \int_0^{\eta_1} \varphi_{11}(\eta)\, d\eta + \int_{\eta_1}^{\eta_2} \varphi_{12}(\eta)\, d\eta; \quad N_2 = -2 \frac{\lambda_2}{\lambda_1} \frac{a_1}{a} \int_0^{\eta_1} d\eta - 2 \frac{a_2}{a} \int_{\eta_1}^{\eta_2} d\eta.$$

Relations (13), (23) were employed to solve the particular heat-conduction problem for a two- layer plate with the following initial data: $\delta_1 = 0.0015$ m; $\delta_2 = 0.0055$ m; $\lambda_1 = 0.207$ W/(m·K); $\lambda_2 = 1.28$ W/(m·K); $a_1 = 0.147 \cdot 10^{-6}$ m²/sec; $a_2 = 0.494 \cdot 10^{-6}$ m²/sec; $T_{01} = T_{02} = 100$°C; $T_w = 20$°C.

The results of the solution are shown in Fig. 1. An analysis of these results allows us to conclude that for Fo ≥ 0.06 the results obtained with the use of one coordinate system differ slightly from those obtained by the method of finite differences. If the latter is considered to give the most accurate temperature values, then the results obtained with the use of the local coordinate system are the least reliable.

To improve the accuracy of the solution, it is necessary to increase the number of approximations. In this case, the efficiency of the method using local coordinate systems will increase in view of the substantial decrease in the volume of calculations. This is related to the fact that the dimensionless coordinate in each layer varies from zero to Δ_i. As a result, the process of evaluation of integrals in calculating the quantities M and N becomes substantially simplified, and with a large number of approximations the main volume of computations involves determination of integrals of this kind.

NOTATION

$\Theta_i = (T_i - T_w)/(T_{0i} - T_w)$, relative excess temperature; $\eta_i = x_i/\delta$, dimensionless coordinate of the i-th layer; $\text{Fo} = \alpha\tau/\delta^2$, Fourier number; $\delta = \delta_1 + \delta_2$, total thickness of the two-layer wall; a, smallest thermal diffusivity a_i (i = 1, 2); $\Delta_i = \delta_i/\delta$, dimensionless thickness of the i-th layer; λ_i (i = 1, 2), thermal conductivities; τ, time; T_{0i} (i = 1, 2), initial temperature; T_w, wall temperature.

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